Extraction of HF transport coefficients in hot-QCD

Andrea Beraudo

INFN - Sezione di Torino

EMMI Rapid Reaction Task Force
12-14 December 2016, GSI

Outline

- The Fokker-Planck equation and the approach to thermal equilibrium
- From Fokker-Planck to Langevin: different discretization schemes
- HQ transport in heavy-ion collisions: the hydrodynamic background
- (HQ transport in heavy-ion collisions: the assigned exercise)

Transport theory: the Boltzmann equation

Time evolution of HQ phase-space distribution $f_Q(t, \mathbf{x}, \mathbf{p})$[1]:

$$\frac{d}{dt} f_Q(t, \mathbf{x}, \mathbf{p}) = C[f_Q]$$

- Total derivative along particle trajectory

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{x}} + \mathbf{F}\frac{\partial}{\partial \mathbf{p}}$$

  Neglecting $\mathbf{x}$-dependence and mean fields: $\partial_t f_Q(t, \mathbf{p}) = C[f_Q]$

- Collision integral:

$$C[f_Q] = \int d\mathbf{k} [\underbrace{w(\mathbf{p}+\mathbf{k}, \mathbf{k}) f_Q(\mathbf{p}+\mathbf{k})}_{\text{gain term}} - \underbrace{w(\mathbf{p}, \mathbf{k}) f_Q(\mathbf{p})}_{\text{loss term}}]$$

  $w(\mathbf{p}, \mathbf{k})$: HQ transition rate $\mathbf{p} \to \mathbf{p} - \mathbf{k}$

[1] For results based on BE see e.g. BAMPS papers and Catania-group studies

Approach to equilibrium in the Boltzmann equation

Momentum exchanges occur with light (thermal) partons i of the plasma.
In the *classical limit* (no Pauli-blocking or Bose-enhancement) one has:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \Big[\underbrace{\overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) f_Q(\mathbf{p}') f_i(\mathbf{p}_1')}_{\text{gain term}} - \underbrace{\overline{w}(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}', \mathbf{p}_1') f_Q(\mathbf{p}) f_i(\mathbf{p}_1)}_{\text{loss;term}} \Big]$$

Approach to equilibrium in the Boltzmann equation

Momentum exchanges occur with light (thermal) partons i of the plasma. In the *classical limit* (no Pauli-blocking or Bose-enhancement) one has:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \Big[\underbrace{\overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) f_Q(\mathbf{p}') f_i(\mathbf{p}_1')}_{\text{gain term}} - \underbrace{\overline{w}(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}', \mathbf{p}_1') f_Q(\mathbf{p}) f_i(\mathbf{p}_1)}_{\text{loss;term}} \Big]$$

From *time-reversal* symmetry one has for the transition probability:

$$\overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) = \overline{w}(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}', \mathbf{p}_1'),$$

hence:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) \Big[f_Q(\mathbf{p}') f_i(\mathbf{p}_1') - f_Q(\mathbf{p}) f_i(\mathbf{p}_1) \Big].$$

Approach to equilibrium in the Boltzmann equation

Momentum exchanges occur with light (thermal) partons i of the plasma. In the *classical limit* (no Pauli-blocking or Bose-enhancement) one has:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \Big[\underbrace{\overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) f_Q(\mathbf{p}') f_i(\mathbf{p}_1')}_{\text{gain term}} - \underbrace{\overline{w}(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}', \mathbf{p}_1') f_Q(\mathbf{p}) f_i(\mathbf{p}_1)}_{\text{loss;term}} \Big]$$

From *time-reversal* symmetry one has for the transition probability:

$$\overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) = \overline{w}(\mathbf{p}, \mathbf{p}_1 | \mathbf{p}', \mathbf{p}_1'),$$

hence:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) \Big[f_Q(\mathbf{p}') f_i(\mathbf{p}_1') - f_Q(\mathbf{p}) f_i(\mathbf{p}_1) \Big].$$

$C[f_Q]$ vanishes *if and only if* $f_Q(\mathbf{p}') f_i(\mathbf{p}_1') = f_Q(\mathbf{p}) f_i(\mathbf{p}_1)$, which entails:

$$f_Q(\mathbf{p}) = \exp\left[-E_{\mathbf{p}}/T\right] \quad \text{and} \quad f_i(\mathbf{p}_1) = \exp\left[-E_{\mathbf{p}_1}/T\right].$$

The Boltzmann equation *always* makes heavy quarks relax to a *thermal distribution* at the same temperature of the medium!

From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[2] (Landau)

$$C[f_Q] \approx \int d\mathbf{k} \left[k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\mathbf{p}, \mathbf{k}) f_Q(t, \mathbf{p})]$$

[2] B. Svetitsky, PRD 37, 2484 (1988)

From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[2] (Landau)

$$C[f_Q] \approx \int d\mathbf{k} \left[k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\mathbf{p},\mathbf{k}) f_Q(t,\mathbf{p})]$$

The Boltzmann equation reduces to the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} f_Q(t,\mathbf{p}) = \frac{\partial}{\partial p^i} \left\{ A^i(\mathbf{p}) f_Q(t,\mathbf{p}) + \frac{\partial}{\partial p^j} [B^{ij}(\mathbf{p}) f_Q(t,\mathbf{p})] \right\}$$

where (verify!)

$$A^i(\mathbf{p}) = \int d\mathbf{k}\, k^i w(\mathbf{p},\mathbf{k}) \longrightarrow \underbrace{A^i(\mathbf{p}) = A(p)\, p^i}_{\text{friction}}$$

$$B^{ij}(\mathbf{p}) = \frac{1}{2} \int d\mathbf{k}\, k^i k^j w(\mathbf{p},\mathbf{k}) \longrightarrow \underbrace{B^{ij}(\mathbf{p}) = \hat{p}^i \hat{p}^j B_0(p) + (\delta^{ij} - \hat{p}^i \hat{p}^j) B_1(p)}_{\text{momentum broadening}}$$

[2] B. Svetitsky, PRD 37, 2484 (1988)

From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[2] (Landau)

$$C[f_Q] \approx \int d\mathbf{k} \left[k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\mathbf{p}, \mathbf{k}) f_Q(t, \mathbf{p})]$$

The Boltzmann equation reduces to the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} f_Q(t, \mathbf{p}) = \frac{\partial}{\partial p^i} \left\{ A^i(\mathbf{p}) f_Q(t, \mathbf{p}) + \frac{\partial}{\partial p^j} [B^{ij}(\mathbf{p}) f_Q(t, \mathbf{p})] \right\}$$

where (verify!)

$$A^i(\mathbf{p}) = \int d\mathbf{k}\, k^i w(\mathbf{p}, \mathbf{k}) \longrightarrow \underbrace{A^i(\mathbf{p}) = A(p)\, p^i}_{\text{friction}}$$

$$B^{ij}(\mathbf{p}) = \frac{1}{2} \int d\mathbf{k}\, k^i k^j w(\mathbf{p}, \mathbf{k}) \longrightarrow \underbrace{B^{ij}(\mathbf{p}) = \hat{p}^i \hat{p}^j B_0(p) + (\delta^{ij} - \hat{p}^i \hat{p}^j) B_1(p)}_{\text{momentum broadening}}$$

Problem reduced to the *evaluation of three transport coefficients*, directly derived from the scattering matrix

---

[2] B. Svetitsky, PRD 37, 2484 (1988)

Approach to equilibrium in the FP equation

The FP equation can be viewed as a continuity equation for the phase-space distribution of the kind $\partial_t \rho(t,\vec{p}) + \vec{\nabla}_p \cdot \vec{J}(t,\vec{p}) = 0$

$$\frac{\partial}{\partial t} \underbrace{f_Q(t,\mathbf{p})}_{\equiv \rho(t,\vec{p})} = \frac{\partial}{\partial p^i} \underbrace{\left\{ A^i(\mathbf{p}) f_Q(t,\mathbf{p}) + \frac{\partial}{\partial p^j} [B^{ij}(\mathbf{p}) f_Q(t,\mathbf{p})] \right\}}_{\equiv - J^i(t,\vec{p})}$$

admitting a steady solution $f_{\rm eq}(p) \equiv e^{-E_p/T}$ when the current vanishes:

$$A^i(\vec{p}) f_{\rm eq}(p) = -\frac{\partial B^{ij}(\vec{p})}{\partial p^j} f_{\rm eq}(p) - B^{ij}(\mathbf{p}) \frac{\partial f_{\rm eq}(p)}{\partial p^j}.$$

One gets

$$A(p) p^i = \frac{B_1(p)}{T E_p} - \frac{\partial}{\partial p^j} \left[\delta^{ij} B_0(p) + \hat{p}^i \hat{p}^j (B_1(p) - B_0(p)) \right],$$

leading to the Einstein *fluctuation-dissipation* relation

$$A(p) = \frac{B_1(p)}{T E_p} - \left[\frac{1}{p} \frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2} (B_1(p) - B_0(p)) \right]$$

The relativistic Langevin equation

The Fokker-Planck equation can be recast into a form suitable to follow the dynamics of each individual quark: the Langevin equation

$$\frac{\Delta p^i}{\Delta t} = -\underbrace{\eta_D(p)p^i}_{\text{determ.}} + \underbrace{\xi^i(t)}_{\text{stochastic}} ,$$

with the properties of the noise encoded in

$$\langle \xi^i(\mathbf{p}_t)\xi^j(\mathbf{p}_{t'})\rangle = b^{ij}(\mathbf{p})\frac{\delta_{tt'}}{\Delta t} \qquad b^{ij}(\mathbf{p}) \equiv \kappa_L(p)\hat{p}^i\hat{p}^j + \kappa_T(p)(\delta^{ij} - \hat{p}^i\hat{p}^j)$$

The relativistic Langevin equation

The Fokker-Planck equation can be recast into a form suitable to follow the dynamics of each individual quark: the Langevin equation

$$\frac{\Delta p^i}{\Delta t} = -\underbrace{\eta_D(p)p^i}_{\text{determ.}} + \underbrace{\xi^i(t)}_{\text{stochastic}} ,$$

with the properties of the noise encoded in

$$\langle \xi^i(\mathbf{p}_t)\xi^j(\mathbf{p}_{t'})\rangle = b^{ij}(\mathbf{p})\frac{\delta_{tt'}}{\Delta t} \qquad b^{ij}(\mathbf{p}) \equiv \kappa_L(p)\hat{p}^i\hat{p}^j + \kappa_T(p)(\delta^{ij} - \hat{p}^i\hat{p}^j)$$

Transport coefficients to calculate:

- *Momentum diffusion*;
- *Friction* term

In the following we will establish their link with the transport coefficients appearing in the Fokker-Planck equation. In particular, the momentum dependence of the noise term requires some care.

Numerical implementation

Introduce the tensor

$$C^{ij}(\mathbf{p}) \equiv \sqrt{\kappa_L(p)}\hat{p}^i\hat{p}^j + \sqrt{\kappa_T(p)}(\delta^{ij} - \hat{p}^i\hat{p}^j) \equiv \sqrt{\kappa_L(p)}P_L^{ij} + \sqrt{\kappa_T(p)}P_T^{ij}$$

and redefine the noise variable as

$$\xi^i \equiv C^{ik}(\mathbf{p})\frac{1}{\sqrt{\Delta t}}\zeta^k \quad \text{with} \quad \langle\zeta^k(t_n)\zeta^l(t_m)\rangle = \delta_{mn}\delta^{kl}.$$

The Langevin equation becomes then

$$\Delta p^i = -\eta_D(p)p^i\Delta t + C^{ik}(\mathbf{p} + \xi\Delta\mathbf{p})\sqrt{\Delta t}\,\zeta^k,$$

where, for the sake of generality, the argument of the strength of the noise term can be evaluated ($\xi \in [0,1]$) at any point in the interval $[\mathbf{p}, \mathbf{p} + \Delta\mathbf{p}]$. In the following we will consider the cases $\xi = 0$ (Ito *pre-point* scheme) and $\xi = 1$ (*post-point* scheme).

The link with the Fokker-Plank equation

Consider an arbitrary function of the HQ momentum $g(\mathbf{p})$ and take the expectation value over the thermal ensemble of its variation, keeping only terms up to order Δt:

$$\langle g(\mathbf{p}+\Delta\mathbf{p}) - g(\mathbf{p})\rangle = \left\langle \frac{\partial g}{\partial p^i}\Delta p^i + \frac{1}{2}\frac{\partial^2 g}{\partial p^i \partial p^j}\Delta p^i \Delta p^j \right\rangle + \dots$$

From

$$\Delta p^i = -\eta_D(p) p^i \Delta t + C^{ik}(\mathbf{p}+\xi\Delta\mathbf{p})\sqrt{\Delta t}\,\zeta^k, \quad \langle\zeta^k\rangle = 0, \quad \langle\zeta^k\zeta^l\rangle = \delta^{kl}$$

one gets:

$$\langle g(\mathbf{p}+\Delta\mathbf{p}) - g(\mathbf{p})\rangle = \left\langle \frac{\partial g}{\partial p^i}\left(-\eta_D p^i + \xi\frac{\partial C^{ik}}{\partial p^j}C^{jk}\right) + \frac{1}{2}\frac{\partial^2 g}{\partial p^i \partial p^j}C^{ik}C^{jk} \right\rangle \Delta t + \dots$$

In the above the expectation value is taken accorrding to the HQ phase-space distribution

$$\langle g(\mathbf{p})\rangle_t \equiv \int d\mathbf{p}\, g(\mathbf{p}) f(t,\mathbf{p})$$

The link with the Fokker-Plank equation

Time evolution given be the differential equation

$$\frac{d}{dt}\langle g(\mathbf{p})\rangle_t \equiv \int d\mathbf{p}\, g(\mathbf{p}) \frac{\partial}{\partial t} f(t,\mathbf{p})$$

Integrating by parts:

$$\mathrm{LHS} = \int d\mathbf{p}\, g(\mathbf{p}) \left\{ \frac{\partial}{\partial p^i} \left[\left(\eta_D p^i - \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk} \right) f(t,\mathbf{p}) \right] + \frac{1}{2} \frac{\partial^2}{\partial p^i \partial p^j} \left[(C^{ik} C^{jk}) f(t,\mathbf{p}) \right] \right\}$$

Comparing with the FP equation

$$\frac{\partial}{\partial t} f_Q(t,\mathbf{p}) = \frac{\partial}{\partial p^i} \left[A^i(\mathbf{p}) f_Q(t,\mathbf{p}) \right] + \frac{\partial}{\partial p^i \partial p^j} [B^{ij}(\mathbf{p}) f_Q(t,\mathbf{p})]$$

one gets

$$A^i(\mathbf{p}) \equiv A(p) p^i = \eta_D p^i - \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

$$C^{ij}(\mathbf{p}) \equiv \sqrt{\kappa_L(p)} P_L^{ij} + \sqrt{\kappa_T(p)} P_T^{ij} = \sqrt{2B_1(p)} P_L^{ij} + \sqrt{2B_0(p)} P_T^{ij}$$

Dependence on the discretization scheme

The transport coefficients describing momentum-diffusion in the Langevin equation *always* coincide with the corresponding ones in the Fokker-Planck equation, no matter which discretization scheme is employed. In general, this is not the case for the friction term. From

$$\eta_D(p)p^i = A(p)p^i + \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

one gets

$$\eta_D(p) = A(p) + \xi \left[\frac{1}{p}\frac{\partial B_1}{\partial p} + \frac{d-1}{p^2}\sqrt{2B_0(p)}(\sqrt{2B_1(p)} - \sqrt{2B_0(p)}) \right]$$

where, furthermore, $A(p)$, $B_0(p)$ and $B_1(p)$ are related by the Einstein relation.

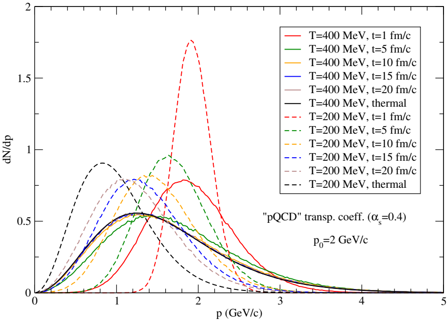
The *pre-point* Ito discretization

Actually, in the Ito *pre-point* scheme $\xi = 0$, so that the friction coeffiecients appearing in the FP and Langevin equations are the same: $A(p) = \eta_D^{\mathrm{pre}}(p)$. Furthermore, in order to approach thermal equilibrium, the Einstein relation must be satisfied:

$$\eta_D^{\mathrm{pre}}(p) = A(p) = \frac{B_1(p)}{TE_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p))\right]$$

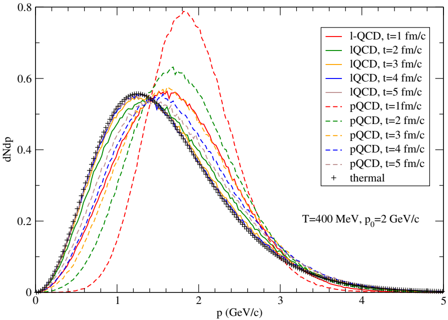
NB: $A(p)$, $B_0(p)$ and $B_1(p)$ can be calculated from the scattering matrix. However, since the Einstein relation must satisfied, one has to calculate only two of them and fix the last one through the above equation

Pre-point scheme: results of the Langevin equation



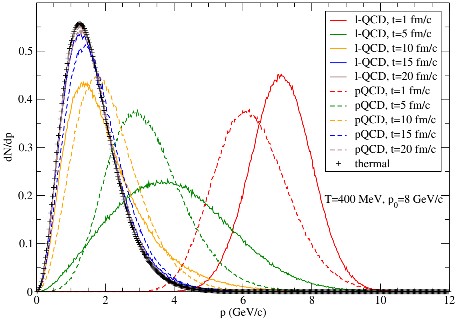
- Starting with a sample of quarks at p_0=2 GeV, at T=400 MeV the particles rapidly approach kinetic equilibrium, while at T=200 MeV sizeable deviations are visible even at late time

Pre-point scheme: results of the Langevin equation



- Starting with a sample of quarks at p_0=2 GeV, at T=400 MeV the particles rapidly approach kinetic equilibrium, while at T=200 MeV sizeable deviations are visible even at late time
- Starting at p_0=2 GeV, the approach to thermal equilibrium is much faster in the case of l-QCD transport coefficients

Pre-point scheme: results of the Langevin equation



- Starting with a sample of quarks at p_0=2 GeV, at T=400 MeV the particles rapidly approach kinetic equilibrium, while at T=200 MeV sizeable deviations are visible even at late time
- Starting at p_0=2 GeV, the approach to thermal equilibrium is much faster in the case of l-QCD transport coefficients
- With p_0=8 GeV, the approach to equilibrium is initially faster for "pQCD" transport coefficients, due to their rise with momentum

The *post-point* discretization

In the *post-point* scheme $\xi = 1$, so that

$$\eta_D^{\rm post}(p)p^i = A(p)p^i + \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

Notice that $\eta_D^{\rm post}(p)$ entering in the Langevin equation *is not* the quantity which is directly evaluated from the scattering matrix!

The *post-point* discretization

In the *post-point* scheme $\xi = 1$, so that

$$\eta_D^{\rm post}(p) p^i = A(p) p^i + \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

Notice that $\eta_D^{\rm post}(p)$ entering in the Langevin equation *is not* the quantity which is directly evaluated from the scattering matrix! Imposing the Einstein relation one has

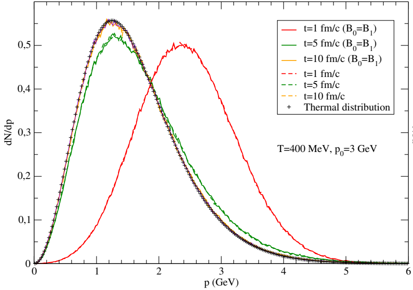
$$\eta_D^{\rm post}(p) = \frac{B_1(p)}{TE_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p))\right] + \left[\frac{1}{p}\frac{\partial B_1}{\partial p} + \frac{d-1}{p^2}\sqrt{2B_0(p)}(\sqrt{2B_1(p)} - \sqrt{2B_0(p)})\right]$$

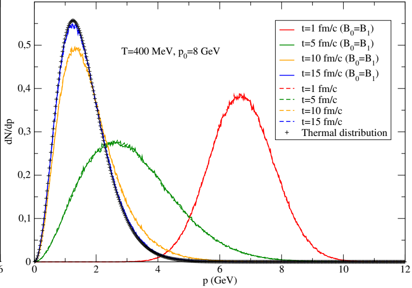
Notice that in the case $B_0(p) = B_1(p) = D(p)$ one has simply

$$\eta_D^{\rm post}(p) = \frac{D(p)}{TE_p}$$

However, in general this *is not* the case and, furthermore, $\eta_D^{\rm post}(p)$ does not follow from any scattering amplitude!

Pre-point scheme: results of the Langevin equation

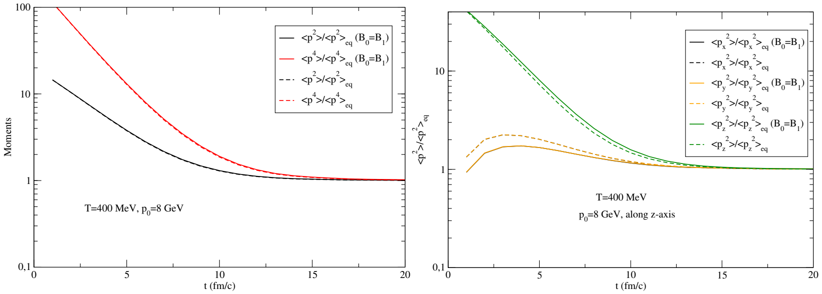




Here we show results obtained with

- $\eta_D^{\rm post}(p)$ tabulated and $D(p) = \eta_D^{\rm post}(p) E_p T$ $(B_0 = B_1)$
- $\eta_D^{\rm post}(p)$ and $B_0(p)$ tabulated and $B_1 = \eta_D^{\rm post}(p) E_p T$ (Einstein relation *not* satisfied!)

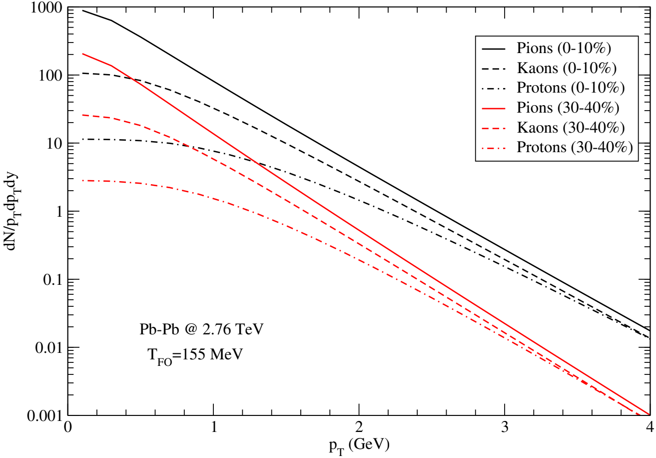
Pre-point scheme: results of the Langevin equation



Here we show results obtained with

- $\eta_D^{\rm post}(p)$ tabulated and $D(p)=\eta_D^{\rm post}(p)E_p T$ $(B_0=B_1)$
- $\eta_D^{\rm post}(p)$ and $B_0(p)$ tabulated and $B_1=\eta_D^{\rm post}(p)E_p T$ (Einstein relation *not* satisfied!)

Hydrodynamic background



Hydrodynamic background